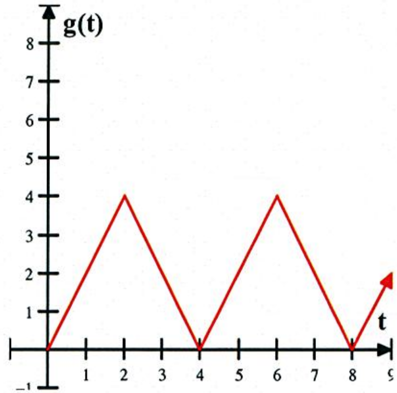
Find the Laplace transform of the periodic function whose graph is shown on the right. SCORE: _____ / 8 PTS

**NOTE: Each "piece" of the function is linear.**




$$g_4(t) = \boxed{\begin{cases} 2t, & 0<t<2 \\ 8-2t, & 2<t<4 \\ 0, & t>4 \end{cases}} = 2t + \begin{cases} 0, & 0<t<2 \\ 8-4t, & 2<t<4 \\ -2t, & t>4 \end{cases}$$

1

$$= 2t + u(t-2)(8-4t) + u(t-4)(2t-8)$$ 1½

$$\mathcal{L}\{g_4(t)\} = \frac{2}{s^2} + e^{-2s}\mathcal{L}\{8-4(t+2)\} + e^{-4s}\mathcal{L}\{2(t+4)-8\}$$

½ 1 1

$$= \frac{2}{s^2} + e^{-2s}\mathcal{L}\{-4t\} + e^{-4s}\mathcal{L}\{2t\}$$

$$= \frac{2}{s^2} - e^{-2s}\left(\frac{4}{s^2}\right) + e^{-4s}\left(\frac{2}{s^2}\right)$$

½ ½

$$\mathcal{L}\{g(t)\} = \frac{\frac{2}{s^2} - e^{-2s}\left(\frac{4}{s^2}\right) + e^{-4s}\left(\frac{2}{s^2}\right)}{1-e^{-4s}}$$ 1

$$= \frac{2-4e^{-2s}+2e^{-4s}}{s^2(1-e^{-4s})}$$

$$= \frac{2(1-e^{-2s})^{\cancel{2}}}{s^2\cancel{(1-e^{-2s})}(1+e^{-2s})}$$

$$= \boxed{\frac{2(1-e^{-2s})}{s^2(1+e^{-2s})}}$$ 1

$$\int_0^4 e^{-st} g(t)\,dt = \int_0^2 2te^{-st}\,dt + \int_2^4 (8-2t)e^{-st}\,dt \qquad (1)$$

ALTERNATE METHOD FOR FIRST 6 POINTS

(1)

$$= \left(-\frac{2}{s}te^{-st} - \frac{2}{s^2}e^{-st}\right)\Bigg|_0^2 \qquad (1)$$

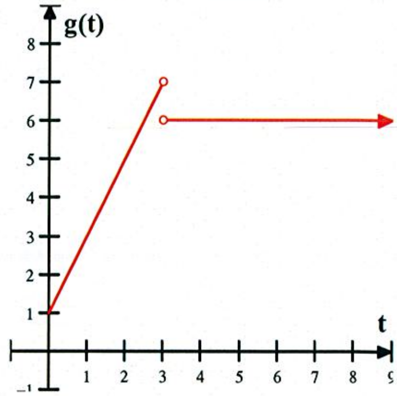
$$+ \left(-\frac{8}{s}e^{-st} + \frac{2}{s}te^{-st} + \frac{2}{s^2}e^{-st}\right)\Bigg|_2^4$$

| $u$ | | $dv$ |
|---|---|---|
| $2t$ | $+$ | $e^{-st}$ |
| $2$ | $-$ | $-\frac{1}{s}e^{-st}$ |
| $0$ | | $\frac{1}{s^2}e^{-st}$ |

$$= \boxed{\begin{aligned}&\cancel{-\frac{4}{s}e^{-2s}} - \frac{2}{s^2}e^{-2s} + \frac{2}{s^2} \\ &\cancel{-\frac{8}{s}e^{-4s}} + \cancel{\frac{8}{s}e^{-4s}} + \frac{2}{s^2}e^{-4s}\end{aligned}} + \cancel{\frac{8}{s}e^{-2s}} \cancel{-\frac{4}{s}e^{-2s}} - \frac{2}{s^2}e^{-2s} \qquad \left(1\tfrac{1}{2}\right)$$

$$= \underbrace{\frac{2}{s^2}}_{\left(\frac{1}{2}\right)} \underbrace{- \frac{4}{s^2}e^{-2s}}_{\left(\frac{1}{2}\right)} \underbrace{+ \frac{2}{s^2}e^{-4s}}_{\left(\frac{1}{2}\right)}$$

Use Laplace transforms to solve the initial value problem $y'' + 4y' + 4y = g(t)$, $y(0) = -3$, $y'(0) = 2$ **SCORE: _____ / 22 PTS**
where $g(t)$ is the function whose graph is shown on the right.




**NOTE: Each "piece" of $g(t)$ is linear.**

$$g(t) = \begin{cases} 1+2t, & 0<t<3 \\ 6, & t>3 \end{cases} = 1+2t + \begin{cases} 0, & 0<t<3 \\ 5-2t, & t>3 \end{cases}$$

$$= 1+2t+u(t-3)(5-2t)$$

$$\mathcal{L}\{g(t)\} = \frac{1}{s} + \frac{2}{s^2} + e^{-3s}\mathcal{L}\{5-2(t+3)\}$$

$$= \frac{1}{s} + \frac{2}{s^2} + e^{-3s}\mathcal{L}\{-1-2t\}$$

$$= \frac{1}{s} + \frac{2}{s^2} - e^{-3s}\left(\frac{1}{s} + \frac{2}{s^2}\right)$$

1 EACH EXCEPT AS OTHERWISE NOTED

$$\begin{aligned} &s^2Y - sy(0) - y'(0) \\ &+4sY \quad -4y(0) \\ &+4Y \end{aligned} = (s^2+4s+4)Y + 3s + 10 = \frac{1}{s} + \frac{2}{s^2} - e^{-3s}\left(\frac{1}{s} + \frac{2}{s^2}\right)$$

$$Y = -\frac{3s+10}{(s+2)^2} + \frac{\cancel{s+2}}{s^2(s+2)^{\cancel{2}}} - e^{-3s}\left(\frac{\cancel{s+2}}{s^2(s+2)^{\cancel{2}}}\right)$$

$$\frac{3s+10}{(s+2)^2} = \frac{3(s+2)+4}{(s+2)^2} = \frac{3}{s+2} + \frac{4}{(s+2)^2}$$ 2

$$\frac{1}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} \longrightarrow As(s+2) + B(s+2) + Cs^2 = 1$$

$s=0$: $2B = 1 \longrightarrow B = \frac{1}{2}$

$s=-2$: $4C = 1 \longrightarrow C = \frac{1}{4}$

$s^2$: $A + C = 0 \longrightarrow A = -\frac{1}{4}$

SANITY CHECK: $s = 2$ $\quad \frac{1}{16} \stackrel{?}{=} \frac{-\frac{1}{4}}{2} + \frac{\frac{1}{2}}{4} + \frac{\frac{1}{4}}{4}$

$= -\frac{1}{8} + \frac{1}{8} + \frac{1}{16}$ ✓

6

$$\mathcal{L}^{-1}\left\{\frac{-\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s^2} + \frac{\frac{1}{4}}{s+2}\right\} = -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t}$$

$$y = -3e^{-2t} - 4te^{-2t} - \frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t} - u(t-3)\left(-\frac{1}{4} + \frac{1}{2}(t-3) + \frac{1}{4}e^{-2(t-3)}\right)$$

$$= -\frac{11}{4}e^{-2t} - 4te^{-2t} - \frac{1}{4} + \frac{1}{2}t - u(t-3)\left(-\frac{7}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2(t-3)}\right)$$